

DISTRIBUTIVE QUOTIENTS

P.L. ROBINSON

ABSTRACT. We note that each lattice L has a unique largest distributive quotient, of which every distributive quotient of L is itself a quotient.

Let L be a lattice, with meet and join denoted by \wedge and \vee respectively. A *congruence* on L is an equivalence relation θ on L that is compatible with meet and join in the sense

$$(a_1, a_2) \in \theta \text{ and } (b_1, b_2) \in \theta \Rightarrow (a_1 \wedge b_1, a_2 \wedge b_2) \in \theta \text{ and } (a_1 \vee b_1, a_2 \vee b_2) \in \theta$$

whenever $a_1, b_1, a_2, b_2 \in L$; the corresponding quotient lattice is the set L/θ of blocks (or equivalence classes) with meet and join well-defined by

$$[a]_\theta \wedge [b]_\theta = [a \wedge b]_\theta, \quad [a]_\theta \vee [b]_\theta = [a \vee b]_\theta$$

whenever $a, b \in L$. A lattice L is *distributive* (notation: $L \in \mathbb{D}$) exactly when it satisfies either (hence each) of the equivalent conditions

$$(\forall a, b, c \in L) \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

$$(\forall a, b, c \in L) \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

The purpose of this brief note is to record certain elementary facts regarding the distributive quotients of an arbitrary lattice: in particular, the fact that each lattice L has a unique largest distributive quotient, of which every distributive quotient of L is itself a quotient.

The congruences on L themselves constitute a lattice $\Theta(L) = \text{Con}(L)$ in which meet is intersection and the join of two congruences is the transitive closure of their union. Within $\Theta(L)$ we single out those congruences of L relative to which the quotient is distributive:

$$\Theta_{\mathbb{D}}(L) = \{\theta \in \Theta(L) : L/\theta \in \mathbb{D}\}.$$

Note that $\Theta_{\mathbb{D}}(L)$ is a filter in $\Theta(L)$: that is, an up-set closed under finite intersections. In fact, more is true.

Theorem 1. $\Theta_{\mathbb{D}}(L) \subseteq \Theta(L)$ is an up-set that is closed under arbitrary intersections.

Proof. $\Theta_{\mathbb{D}}(L)$ is an up-set: if $\Theta_{\mathbb{D}}(L) \ni \theta_0 \subseteq \theta \in \Theta(L)$ then the inclusion $\theta_0 \subseteq \theta$ induces a surjective homomorphism $L/\theta_0 \rightarrow L/\theta$ realizing L/θ as a quotient of the distributive lattice L/θ_0 ; thus $L/\theta \in \mathbb{D}$ and so $\theta \in \Theta_{\mathbb{D}}(L)$. $\Theta_{\mathbb{D}}(L)$ is closed under arbitrary intersections: if $\theta_\lambda \in \Theta_{\mathbb{D}}(L)$ for each $\lambda \in \Lambda$ then the canonical map

$$L \rightarrow \prod_{\lambda} (L/\theta_\lambda) : a \mapsto ([a]_{\theta_\lambda})_\lambda$$

factors through an injective homomorphism

$$L/(\cap_{\lambda} \theta_\lambda) \rightarrow \prod_{\lambda} (L/\theta_\lambda) \in \mathbb{D};$$

thus $L/(\cap_{\lambda} \theta_\lambda) \in \mathbb{D}$ and so $\cap_{\lambda} \theta_\lambda \in \Theta_{\mathbb{D}}(L)$. □

Thus, $\Theta_{\mathbb{D}}(L)$ has as least element its infimum

$$\delta_L = \bigwedge \Theta_{\mathbb{D}}(L) = \bigcap \{\theta : \theta \in \Theta_{\mathbb{D}}(L)\} \in \Theta_{\mathbb{D}}(L)$$

and so $\Theta_{\mathbb{D}}(L)$ is principal with δ_L as generator:

$$\Theta_{\mathbb{D}}(L) = \uparrow \delta_L = \{\theta \in \Theta(L) : \theta \supseteq \delta_L\}.$$

Observe that L/δ_L is the largest distributive quotient of L : in fact, if L/θ is any distributive quotient of L then $\theta \in \Theta_{\mathbb{D}}(L) = \uparrow \delta_L$; thus, θ contains δ_L and so there is a canonical surjective homomorphism $L/\delta_L \rightarrow L/\theta$.

We may identify the generator δ for an arbitrary quotient as follows.

Theorem 2. *If $\theta \in \Theta(L)$ then $\delta(L/\theta) = \delta_L \vee \theta/\theta$.*

Proof. For convenience, write $\delta = \delta_L$. On the one hand, the isomorphism

$$(L/\theta)/(\delta \vee \theta/\theta) \cong L/(\delta \vee \theta) \in \mathbb{D}$$

places the congruence $\delta \vee \theta/\theta$ in $\Theta_{\mathbb{D}}(L/\theta)$. On the other hand, let ϕ be a congruence of L containing θ : if $\phi/\theta \in \Theta_{\mathbb{D}}(L/\theta)$ then the isomorphism

$$L/\phi \cong (L/\theta)/(\phi/\theta) \in \mathbb{D}$$

forces $\phi \in \Theta_{\mathbb{D}}(L)$ so ϕ also contains δ and $\phi \in \uparrow (\delta \vee \theta)$. Conclusion: $\Theta_{\mathbb{D}}(L/\theta) = \uparrow (\delta \vee \theta/\theta)$. \square

We may identify the generator δ for a finite product as follows.

Theorem 3. $\delta(L_1 \times L_2) = \delta_{L_1} \times \delta_{L_2}$.

Proof. For convenience, write $\delta_1 = \delta_{L_1}$ and $\delta_2 = \delta_{L_2}$. On the one hand, the isomorphism

$$(L_1 \times L_2)/(\delta_1 \times \delta_2) \cong (L_1/\delta_1) \times (L_2/\delta_2) \in \mathbb{D}$$

places $\delta_1 \times \delta_2$ in $\Theta_{\mathbb{D}}(L_1 \times L_2)$. On the other hand, each $\theta \in \Theta(L_1 \times L_2)$ has the form $\theta_1 \times \theta_2$ for $\theta_1 \in \Theta(L_1)$ and $\theta_2 \in \Theta(L_2)$; now, if $\theta \in \Theta_{\mathbb{D}}(L_1 \times L_2)$ then

$$(L_1/\theta_1) \times (L_2/\theta_2) \cong (L_1 \times L_2)/\theta \in \mathbb{D}$$

forces $(L_1/\theta_1) \in \mathbb{D}$ and $(L_2/\theta_2) \in \mathbb{D}$ so that $\theta_1 \in \Theta_{\mathbb{D}}(L_1) = \uparrow \delta_1$ and $\theta_2 \in \Theta_{\mathbb{D}}(L_2) = \uparrow \delta_2$ whence $\theta = \theta_1 \times \theta_2 \in \uparrow (\delta_1 \times \delta_2)$. Conclusion: $\Theta_{\mathbb{D}}(L_1 \times L_2) = \uparrow (\delta_1 \times \delta_2)$. \square

Let us identify the generator δ_L of $\Theta_{\mathbb{D}}(L)$ for a lattice L in some basic examples.

Example 0. If L is a distributive lattice then, as each quotient of L is distributive, $\Theta_{\mathbb{D}}(L) = \Theta(L)$ and $\delta_L = \underline{0}$ is the equality (or diagonal) relation on L .

Example 1. The non-distributive ‘diamond’ M_3 is simple; it follows at once that $\Theta_{\mathbb{D}}(M_3) = \{\underline{1}\}$ so that $\delta_{M_3} = \underline{1} = M_3 \times M_3$ is the trivial congruence that fully collapses M_3 .

Example 2. The non-modular ‘pentagon’ $N_5 = \{0, a, b, c, 1\}$ with $a > b$ yields a distributive quotient as soon as a and b are identified. Accordingly, δ_{N_5} is the principal congruence $\theta(a, b)$: that is, the smallest congruence containing the pair (a, b) ; its only nontrivial block is the doubleton $\{a, b\}$.

Example 3. The free modular lattice $F_{\mathbb{M}}(3)$ on three generators x, y, z admits a unique homomorphism to the free distributive lattice $F_{\mathbb{D}}(3)$ on x, y, z respecting the generators; the kernel of this homomorphism is the principal congruence $\theta(u, v)$ that identifies $u = (y \vee z) \wedge (z \vee x) \wedge (x \vee y)$ and $v = (y \wedge z) \vee (z \wedge x) \vee (x \wedge y)$. As no smaller congruence can yield a distributive quotient, $\delta_{F_{\mathbb{M}}(3)} = \theta(u, v)$. The nontrivial blocks of this congruence are six doubletons and the diamond with top u and bottom v .

Example 4. The case of the free lattice $F(n)$ on n generators is similar: $\delta_{F(n)}$ is the kernel of the unique homomorphism $F(n) \rightarrow F_{\mathbb{D}}(n)$ that respects all n generators.

Remark 1. We have considered only the class \mathbb{D} of distributive lattices, but entirely similar considerations apply to the class \mathbb{M} of modular lattices. Indeed, they apply to any equational class \mathbb{K} of lattices: as \mathbb{K} is closed under the formation of quotients, products and sublattices, if L is any lattice then the filter

$$\Theta_{\mathbb{K}}(L) = \{\theta \in \Theta(L) : L/\theta \in \mathbb{K}\}$$

is principal with generator

$$\kappa_L = \bigwedge \Theta_{\mathbb{K}}(L) = \bigcap \{\theta : \theta \in \Theta_{\mathbb{K}}(L)\} \in \Theta_{\mathbb{K}}(L).$$

In particular, L has a largest quotient L/κ_L in any equational class \mathbb{K} and each quotient of L in \mathbb{K} is actually a quotient of L/κ_L .

Remark 2. This offers a perspective on the free lattice $F_{\mathbb{K}}(P)$ over the class \mathbb{K} generated by the poset P as discussed in section 5 of [1]: thus, let $F(P)$ be the free lattice generated by P as in [1] Corollary 5.7; the lattice $F_{\mathbb{K}}(P)$ arises as the quotient of $F(P)$ modulo the congruence $\kappa_{F(P)}$ when the elements of $P \subseteq F(P)$ lie in different blocks of $\kappa_{F(P)}$.

REFERENCES

- [1] G. Grätzer, *Lattice Theory: First Concepts and Distributive Lattices*, W.H. Freeman and Company (1971); Dover Publications (2009).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE FL 32611 USA
E-mail address: paulr@ufl.edu